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A mathematical description of transient crack growth behaviour in glass*

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Transient crack growth behaviour resulting from time-dependent changes in crack-tip radius can occur near the fatigue limit. In the present work, mathematical expressions describing this transient behaviour are developed assuming that a dissolution reaction is responsible for changes in crack geometry. An elliptical crack is analysed because of its mathematical simplicity. The theoretical model slightly underestimates the extent of crack-tip blunting occurring below the fatigue limit. However, the predicted transient behaviour associated with the crack-tip sharpening processes which take place above the fatigue limit compares favourably with experimental data for glass.

1. Introduction

A time-dependent strength reduction is often observed in ceramic or glass specimens which are subjected to mechanical stresses in moist environments [1–3]. This behaviour, which is attributed to stress-assisted or slow crack growth, is typically represented by plotting the time-to-failure, t_f, of a statically loaded sample as a function of the applied stress σ_a (static fatigue). Alternatively, one may directly measure the subcritical crack velocity, V, as a function of the stress intensity factor K_I. An examination of typical σ_a–t_f and V–K_I data for glass [4–6] often provides evidence of a critical σ_a or K_I value below which strength degradation due to stress corrosion does not occur. This critical point, which is called the stress corrosion or static fatigue limit, is of considerable interest for design applications.

Although several mathematical models describing slow crack growth behaviour have been developed, only a limited number have adequately dealt with the threshold region. One of the most familiar is that of Charles and Hillig [7]. In their treatment, slow crack growth was attributed to a stress-enhanced chemical dissolution reaction occurring along the crack surface. They assumed that strength degradation did not occur when the rate of change of the crack-tip stress, σ, was zero. This concept can be illustrated mathematically using the Inglis approximation [8],

$$\sigma = 2\sigma_a(a/\rho)^{1/2} \tag{1}$$

where σ_a is the applied stress, α is the crack length, and ρ is the crack-tip radius. Taking the time derivative of both sides yields

$$\frac{d\sigma}{dt} = \sigma_a(a/\rho)^{1/2}\left[\frac{d(\ln a)}{dt} - \frac{d(\ln \rho)}{dt}\right] \tag{2}$$

where σ_a is taken as constant. When σ_a is above the threshold value, the first term in brackest dominates; $d(\ln a)/dt \gg d(\ln \rho)/dt$, and the crack-tip stress depends only upon a. Since $d(\ln a)/dt > 0$, Equation 2 predicts that σ will increase with time, with failure occurring when the cohesive strength is reached. However, near the fatigue limit, $d(\ln a)/dt$ is comparable with $d(\ln \rho)/dt$ such that $d\sigma/dt \to 0$, and no strength degradation occurs. This variability of the crack-tip radius near the threshold region has not been widely recognized.

The recent crack-growth study by Michalske [9] has done much to elucidate the transient nature of crack-tip blunting and sharpening processes in glass. In his experiment, soda-lime-silica dcb (double cantilever beam) specimens, initially

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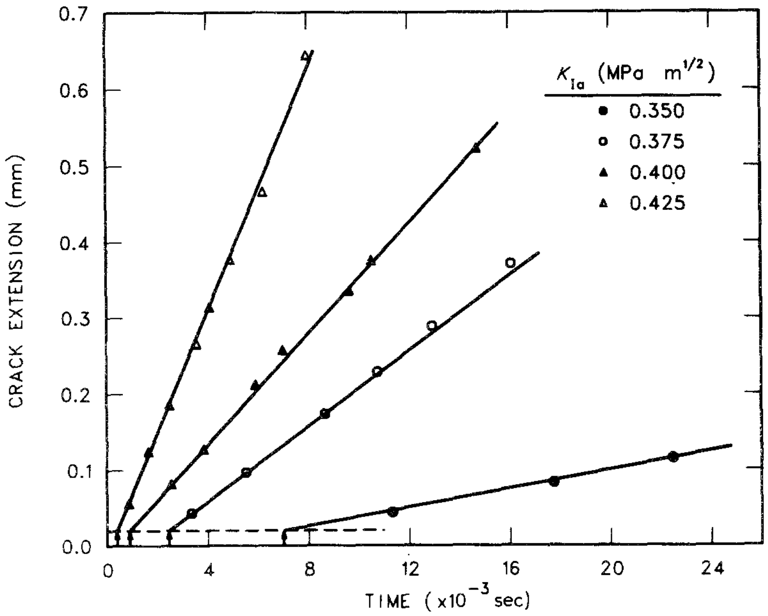

Figure 1 Experimental crack growth data for glass showing transient behaviour following ageing below fatigue limit [9]. Arrows indicate points at which ρ reaches limiting value (defined by dotted line).

containing a sharp crack, were loaded to a K_I value below the threshold stress intensity factor, K_{Ith}, determined from well-established V–K_I data. Following a 0 to 16 h ageing period in a water environment, the samples were reloaded to K_{Ia} ($K_{Ia} > K_{Ith}$) and the crack length measured as a function of time. The results, which are reproduced in Fig. 1, revealed that a finite time was generally required for the crack velocity to reach its steady-state value as dictated by the V–K_I data. The length of the time delay was proportional to both the ageing time and the inverse of K_{Ia}. The author attributed this transient behaviour to the time required for the crack tip, which had become blunted during the ageing step, to reach its steady-state shape*. However no rigorous mathematical model was proposed.

In this paper, a simple expression describing the blunting process for $K_I < K_{Ith}$ is first derived. The resulting information is then used to predict the dependence of crack length on time as K_I is raised above K_{Ith}. Finally, the theoretical curves are compared with existing data.

2. Mathematical treatment

We begin by assuming that the crack configuration can be adequately described by elliptical geometry (Fig. 2). Since for this configuration

$$\rho = b^2/a \tag{3}$$

where b is the length of the minor axis, the crack geometry can be completely described by ρ and a. These two parameters also dictate the stress level at the crack tip in accordance with Equation 1.

Next we assume that the time-dependent changes in ρ, which result from the chemical interaction between the environment and crack surface, are controlled by two competitive processes, i.e.

$$\frac{\partial \rho}{\partial t} = V_0 + \left(\frac{\partial \rho}{\partial a}\right)\left(\frac{\partial a}{\partial t}\right) \tag{4}$$

The quantity V_0 is representative of the corrosion (dissolution) rate of the crack surface located away from the highly stressed crack-tip region (Fig. 2). It is assumed that this rate is approximately constant and comparable with the surface recession rates exhibited by glass specimens aged in water. Estimates of V_0, which can be obtained from experimentally measured dissolution rates of glass in water [10, 11], range from $\sim 1 \times 10^{-14}$ m sec^{-1} to $\sim 2 \times 10^{-15}$ m sec^{-1}.† The latter value was used to approximate V_0 in all subsequent calculations.

The second quantity in Equation 4, $(\partial \rho/\partial a)(\partial a/\partial t)$, represents the effect of crack length upon ρ. It it is assumed that $b \sim$ constant, then the

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*In glass the steady-state ρ is estimated to be approximately 5.0×10^{-10} m.

†Dissolution rates given in [1] and [11] were adjusted to reflect conditions at 25° C. An activation energy of 68.5 kJ mol^{-1} (16.1 kcal mol^{-1}) was used in the associated calculation.

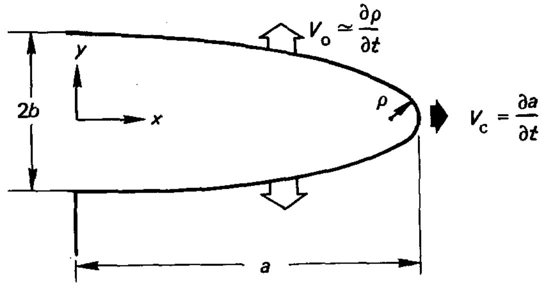

Figure 2 Schematic representation of crack geometry and associated corrosion processes. Elliptical crack changes in ρ are dictated by the competitive corrosion rates V_0 and V_c.

TABLE I Crack growth parameters calculated from V–K_I behaviour of soda-lime-silica glass exposed to water [6]

c_1 (m sec^{-1})	2.906×10^{-14}
c_2 (Pa)$^{-1}$	7.265×10^{-10}
c_3	1069.334
c_4 (m)	5.360×10^{-7}
c_5 (Pa)$^{-1}$	1.803×10^{-10}

differentiation of Equation 3 yields: $(\partial\rho/\partial a) = -(\rho/b)^2$. This is a reasonable assumption when $K_I > K_{Ith}$, since the variation of a with t is orders of magnitude greater than that for b; when $K_I < K_{Ith}$, $\partial a/\partial t \to 0$ (as discussed below) and ρ depends only upon V_0.

The time dependence of a arises primarily from the stress-assisted bond breaking occurring at the crack-tip. There is evidence [12, 13] that the mechanisms responsible for this behaviour are quite different than those controlling the simple dissolution rate V_0. A suitable expression for $\partial a/\partial t$ can be obtained by modifying the theoretical V–K_I relationship developed by Brown [12],

$$\frac{\partial a}{\partial t} = V_c(\rho, \sigma) = [c_1 \exp(c_2\sigma)]\Phi \qquad (5a)$$

$$\Phi = \{1 - \exp[-c_3 + (c_4/\rho) - c_5\sigma]\} \qquad (5b)$$

where V_c is the corrosion rate at the crack tip and c_1 to c_5 are constants characterising the slow crack growth process. This expression is derived by assuming that the bond-rupturing process is thermally activated and thus can be described using reaction rate theory [14]. In this respect, Φ represents the overall driving force for crack growth. When steady-state crack growth conditions prevail, ρ is constant (5.0×10^{-10}m) and V_c depends only upon σ and thus K_I, [$K_I = 1/2\sigma(\pi\rho)^{1/2}$]. At the fatigue limit, $K_I = K_{Ith}$ so that Φ and thus V_c equal zero. The crack growth constants used in the present study (Table I) were determined by fitting the V–K_I data (at 25° C) of Wiederhorn and Bolz [6] to Equation 5.

When $K_I < K_{Ith}$, V_c approaches zero, and Equation 4 can be integrated to give

$$\rho_a - \rho_0 = V_0 t_a \qquad (6)$$

where ρ_a is the radius following the ageing period t_a and ρ_0 is the value prior to the ageing treatment. For $t_a = 5.76 \times 10^4$ sec (16 h), Equation 6 predicts that ρ will increase by 23% to 6.15×10^{-10} m. This quantity compares favourably with the 50% value calculated by Michalske [9]. Reasons for the slight discrepancy are discussed later.

If, following the ageing period, the stress intensity factor is raised above the threshold to K_{Ia}, the crack velocity V_c will be below its steady-state value due to the blunting process. Since V_c is typically larger than V_0 (even for the blunted crack, Equation 4 predicts that the crack-tip radius will gradually decrease until the limiting value of 5×10^{-10}m is reached. The time t_s required for completion of this sharpening process can be calculated by rearranging Equation 4 and then integrating both sides. The result is

$$t_s = \int_{\rho_a}^{\rho_0} [V_0 - (\rho/b)^2 V_c]^{-1}\, d\rho \qquad (7)$$

where ρ_0 and ρ_a have values of 5.0×10^{-10}m and 6.15×10^{-10}m, respectively. Since K_{Ia}, which is a known quantity, equals $1/2\,\sigma(\rho\pi)^{1/2}$, σ depends only on ρ so that V_c can be expressed entirely as a function of ρ (see Equation 5). Therefore, the integral in Equation 7 can be readily determined using numerical techniques. The corresponding plots of crack length against time are shown for various K_{Ia} values in Fig. 3. The dotted portion of each curve represents the transient period during which ρ decreases to its limiting value. During this time V_c gradually increases from its value at $\rho = \rho_a$ to the steady-state velocity ($\rho = \rho_0$).

A comparison between Figs. 1 and 3 reveals fairly good agreement between experimental and predicted behaviour. Unfortunately, a thorough comparison is not possible due to the lack of experimental data in the region associated with the transient behaviour. The absence of data primarily reflects the experimental difficulties involved in measuring small changes in crack length over short

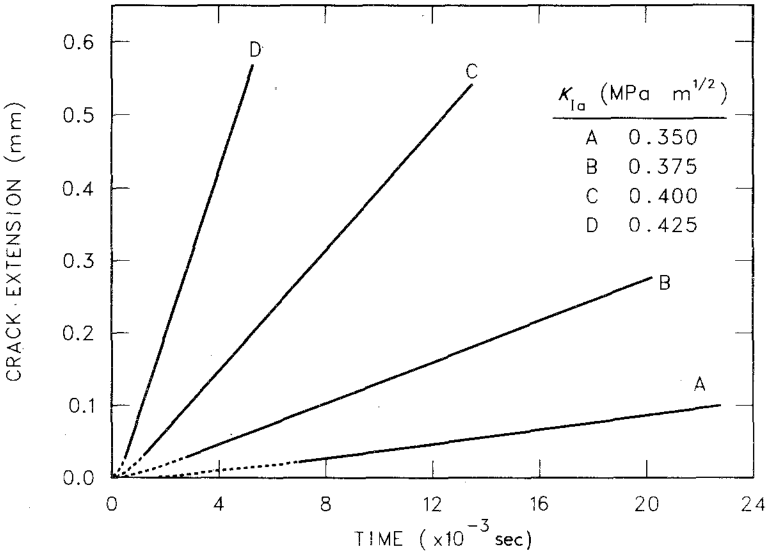

Figure 3 Crack growth behaviour predicted from Equation 7. The dotted lines indicate the transient period associated with sharpening processes.

time intervals. Consequently, the data in Fig. 1 represent crack growth behaviour after steady-state conditions are established. Nevertheless, estimates of the transient time interval t_s can still be obtained from the experimental results by assuming that steady-state conditions prevail when the crack extension exceeds 0.02 mm†. In this case t_s is equal to the x-intercept obtained by extrapolating the crack growth data in Fig. 1 to the horizontal line, $a = 0.02$ mm (dotted line). Table II gives the experimental and predicted values of t_s as well as the corresponding steady-state crack velocities. There is excellent agreement between the respective t_s quantities. However, the experimental and predicted V_c values exhibit rather significant variations, which primarily reflect uncertainties in the crack growth parameters in Table I.

3. Discussion

In the present model, the time-dependent changes in ρ (i.e., blunting and sharpening processes) are solely determined (via Equation 4) by the sum of two competitive chemical processes described by the two rates: V_0 and V_c. The quantity V_0 represents the rate of surface corrosion in the absence of any significant crack-tip stresses. When K_I is less than the experimental threshold, V_0 dominates the crack-tip blunting occurs. However, the lack of complete agreement between predicted and experimental values of the blunted radius ρ_a suggests that the assumption of a uniform surface corrosion (implied by the use of a constant V_0 value) may not be totally valid. For example, it is likely that V_0 depends upon both the radius and local stress. Therefore, improvements in the predicted ρ_a value at the crack tip might be obtained by analysing the corrosion (dissolution) rate at each point along the crack front. In particular, there is recent evidence [15] that enhanced blunting can occur slightly below the stress intensity threshold. Unfortunately, an exact solution to this problem is complicated by the fact that both ρ and σ can vary along the crack surface and, thus, is outside the scope of the present paper.

TABLE II Experimental and predicted values of t_s and V_c

K_{Ia} (MPa m$^{1/2}$)	Experimental values		Predicted values	
	t_s (ksec)	V_c (m sec^{-1})*	t_s (ksec)	V_c (m sec^{-1})*
0.350	7.01	6.27×10^{-9}	7.00	4.40×10^{-9}
0.375	2.50	2.48×10^{-8}	2.90	1.43×10^{-8}
0.400	0.93	3.75×10^{-8}	1.23	4.23×10^{-8}
0.425	0.89	8.42×10^{-8}	0.52	1.18×10^{-7}

*Steady-state values.

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†This criterion is based upon the results predicted from Equation 7.

The elliptical crack model provides fairly accurate predictions of the crack length–time behaviour (including the K_{Ia} dependence) associated with the sharpening process ($K_{Ia} > K_{Ith}$). For this case, V_c is substantially greater than V_0 so that the dissolution rate occurring away from the crack tip has relatively minor effects upon the resulting transient behaviour. In particular, the V_0 term in Equation 7 can be varied by several orders of magnitude without significantly changing t_s.

The transient behaviour illustrated in Fig. 3 is quite similar to that observed in a previous study of the effects of electric field upon crack growth [16, 17]. In this study, the application of a substantial electric field across a crack moving under the influence of an applied K_I, caused the crack to slow, and in some instances stop completely. Upon removal of the field, a transient period on the order of 1 to 10 ksec was required for the crack velocity to reach its steady-state value. This suggests that the crack tip experienced a blunting process during the application of the field similar to that described in the present paper.

Finally, it must be emphasized that the proposed model is subject to several limitations. First, the assumption of an elliptical crack implies that the crack surface is continuous, and thus does not account for atomic discreteness. In particular, it is unlikely that use of a crack tip radius which is of the order of the atomic dimensions has any physical significance. Although more sophisticated and realistic crack geometries are possible, the associated mathematics often prove intractable. A second limitation is that the theoretical model is based on the assumption that a dissolution reaction is totally responsible for time-dependent changes in b and a. However, recent strength studies of a high silica glass aged in water [18] indicate that crack-tip blunting may actually involve dissolution and subsequent precipitation of material. The resulting rate at which blunting occurs is faster than that for the simple dissolution mechanism. This in part may explain why Equation 6 underestimates the extent of the crack-tip rounding.

References

1. S. M. WIEDERHORN, *Int. J. Fract. Mech.* **4** (1968) 171.
2. *Idem,* in "Fracture Mechanics of Ceramics", Vol. 2, Edited by R. Bradt, D. P. Hasselman and F. F. Lange (Plenum, New York, 1974) pp. 613–646.
3. J. B. WACHTMAN, Jr., *J. Amer. Ceram. Soc.* **57** (1974) 509.
4. B. J. S. WILKINS and R. DUTTON, *ibid.* **59** (1976) 108.
5. R. J. CHARLES, *J. Appl. Phys.* **29** (1958) 1549.
6. S. M. WIEDERHORN and L. H. BOLZ, *J. Amer. Ceram. Soc.* **53** (1970) 543.
7. W. B. HILLIG and R. J. CHARLES, "High-Strength Materials," edited by V. F. Zackay (John Wiley and Sons, New York, 1965) pp. 681–705.
8. C. F. INGLIS, *Trans. Inst. Naval Architecture, London,* **55** (1913) 219.
9. T. A. MICHALSKE, in "Fracture Mechanics of Ceramics," Vol. 5, edited by R. C. Bradt, A. G. Evans, D. P. H. Hasselman and F. F. Lange (Plenum Press, New York, 1983) pp. 277–290.
10. S. ITO and M. TOMOZAWA, *Comm. Amer. Ceram. Soc.* **11** (1981) 160.
11. D. HUBBARD and E. H. HAMILTON, *J. Res. Nat. Bur. Stand.* **27** (1941) 143.
12. S. D. BROWN, *J. Amer. Ceram. Soc.* **62** (1979) 515.
13. T. A. MICHALSKE and S. W. FRIEMAN, *ibid.* **66** (1983) 284.
14. H. EYRING and E. M. EYRING, in "Modern Chemical Kinetics," edited by H. H. Sisler and C. A. Vanderwerf (Reinhold, New York, 1962) Ch. 4.
15. E. R. FULLER, private communication (1983).
16. M. G. SCHWABEL and V. D. FRECHETTE, Paper 36-B-78, Presented at the 80th Annual Meeting of the American Ceramic Society, Detroit, Michigan, May 6–11, 1978, *Amer. Ceram. Soc. Bull.* **57** (1978) 307.
17. M. G. SCHWABEL, Ph.D. Thesis, New York State College of Ceramics, Alfred University, Alfred, NY (1978).
18. S. ITO and M. TOMOZAWA, *J. Amer. Ceram. Soc.* **65** (1982) 368.

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